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ON THERMAL CONVECTION BETWEEN NON-UNIFORMLY HEATED PLANES

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Abstract— The stability of natural convection in a thin, horizontal layer subjected to horizontal as well as vertical temperature gradients is investigated on the basis of linear theory. The boundaries are taken to be stress-free and perfectly conducting and the horizontal temperature gradient is assumed to be small. The analysis shows that the critical Rayleigh number is always larger than that for the ordinary Bénard problem. The preferred mode of disturbance is stationary, and will be a transverse roll (having axes normal to the basic flow) or a longitudinal roll (having axes aligned in the direction of the basic flow) depending on whether the Prandtl number is less or larger than 5·1. Finally, some calculations are made of the converted energy associated with the unstable perturbations, indicating that the mechanism of instability is of thermal (convective) origin.

NOMENCLATURE

d, depth of layer;
k, m, wave numbers in the x and z direction;
$\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$, unit vectors;
$\boldsymbol{v}$, velocity vector;
u, v, w, velocity components;
t, time;
$U(y)$, basic flow velocity;
p, pressure;
$P(x, y)$, basic flow pressure;
T, temperature;
T_0, standard temperature;
ΔT, temperature difference between lower and upper plane;
∇^2, Laplacian operator;
L, operator defined by (4.3);
K, I, defined by (5.2) and (5.6), respectively;
Pr, Prandtl number ν/κ;
Ra, Rayleigh number $g\bar{\alpha}\Delta T d^3/\kappa\nu$

(g = acceleration of gravity, $\bar{\alpha}$ = coefficient of volume expansion).

Greek letters
α, overall wave number;
β, horizontal temperature gradient;
θ, Θ, temperatures;
κ, thermal diffusivity;
ν, kinematic viscosity;
ρ, density;
ρ_0, standard density;
σ, amplification factor of disturbance;
ξ, ζ, defined by (4.9) and (4.27), respectively.

Superscripts
$'$, non-dimensional quantities;
$\hat{}$, perturbation quantities.

1. INTRODUCTION

THERMAL convection in thin, horizontal fluid layers uniformly heated from below is quite well described in the literature (the Bénard problem); see the review article by Brindley [1] for references. In many practical problems, however, non-uniformly heating may occur, and thus the layer will be subjected to horizontal as well as vertical temperature variations. Few theoretical attempts have been made to analyse the stability of thin layers under such conditions. Zierep [2] has approached the problem by investigating a model with a discontinuous jump in the bottom temperature. Further, Koschmieder [3] has

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performed a laboratory experiment on convection between circular planes, the upper at constant temperature, the lower non-uniformly heated. At subcritical conditions a density gradient roll was observed, breaking up into axially symmetric rolls of different sizes and rotation when the vertical temperature difference was sufficiently increased. Theoretically Müller [4] has given a two-dimensional linear analysis of this problem. A closely similar experiment in a rectangular cavity has been reported by Berkovsky and Fertman [5].

In the present paper we investigate the effect of horizontal temperature variation on ordinary Bénard convection, assuming that the fluid is unlimited in the lateral directions. Due to the horizontal density gradient thus produced, a shear flow develops, and when the temperature difference between the bottom and top plane exceeds a certain critical value, this flow becomes unstable.

It is well known that in the absence of shear, a non-linear analysis must be applied to obtain the final flow structure, being two-dimensional rolls if the fluid properties are constant (Schlüter *et al.* [6]), or hexagons if the properties vary with temperature (Palm [7], Segel and Stuart [8], Busse [9]).

In stability problems involving a basic flow, a preferred direction is introduced into the system, and a unique flow pattern may be predicted from linear theory (Liang and Acrivos [10]). The selected type of disturbance will depend on the instability mechanisms involved. For non-stratified shear flows, the mechanism is purely hydrodynamical, and by Squire's theorem it can be proved that instability first occurs for rolls having axes normal to the mean flow (transverse rolls). For shear flow with unstable vertical stratification due to heating from below, the instability will be of thermal origin if the basic flow Reynolds number is sufficiently small, and then rolls having axes aligned in the direction of the mean flow (longitudinal rolls) will be preferred [10–12].

In the present problem we shall assume a small horizontal temperature variation which implies a small basic flow velocity. For moderate (or large) Prandtl numbers then, the hydrodynamical instability mechanism will not seriously affect the problem and thermal instability will dominate. Hence longitudinal rolls would be expected. It is therefore a little surprising, at least to the author, that the final flow pattern may be transverse or longitudinal rolls depending on whether the Prandtl number is smaller or larger than 5·1.

The reason for this, however, is purely of thermal origin. This is indicated in the last part of the paper where we consider the conversion of energy between the mean flow and the perturbation. There we show that the horizontal transfer of vertical momentum cannot account for the change of mode about $Pr = 5{\cdot}1$, while the release of potential energy may do so.

2. BASIC FLOW

Consider natural three-dimentional convection of a viscous fluid confined between horizontal planes, see Fig. 1. For mathematical simplicity we shall assume the planes to be stress-free and perfectly conducting, and the lateral temperature variation to be linear in the x-direction. For a given x-coordinate, the temperature difference between the planes is constant, ΔT, and the lower plane is the warmer. We then may write $T = T_0 - \Delta T/2 - \beta x$ and $T = T_0 + \Delta T/2 - \beta x$ at the top and bottom plane, respectively, where β is a positive constant.

To avoid infinite temperatures on the boundaries, we must limit the model in the x-direction, but we assume that the ratio of the depth to the characteristic horizontal dimension is so small that the lateral boundaries will not affect the motion.

Introducing non-dimensional (primed) quantities by

$$(x, y, z) = (x', y', z')\, d\,, \qquad t = t' \frac{d^2}{\kappa}$$

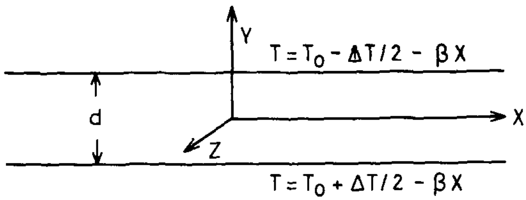

FIG. 1. The coordinate system

$$(u, v, w) = (u', v', w')\frac{\kappa}{d}, \qquad p = p'\frac{\rho_0 \kappa v}{d^2}$$

$$T - T_0 = T'\Delta T$$

and making the Boussinesq approximation, we write the governing equations in vector notation

$$Pr^{-1}\left(\frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v}.\nabla\boldsymbol{v}\right) = -\nabla p + \nabla^2\boldsymbol{v} + RaT\boldsymbol{j} \quad (2.1)$$

$$\frac{\partial T}{\partial t} + \boldsymbol{v}.\nabla T = \nabla^2 T \quad (2.2)$$

$$\nabla.\boldsymbol{v} = 0 \quad (2.3)$$

where the primes have been dropped.

We now consider a particular solution of these equations. Setting

$$\frac{\partial}{\partial t} = v = w = 0$$

$$u = U(y) \quad (2.4)$$

$$T = T(y) - \beta x$$

where β now is dimensionless, and eliminating the pressure from (2.1), the governing equations reduce to

$$D^3U(y) = -\beta Ra$$

$$D^2T(y) = -\beta U \quad (2.5)$$

where

$$D = \frac{d}{dy}.$$

These are subject to the boundary conditions

$$DU(\pm\tfrac{1}{2}) = 0, \qquad T(\pm\tfrac{1}{2}) = \mp\tfrac{1}{2} \quad (2.6)$$

and to the continuity condition

$$\int_{-\frac{1}{2}}^{+\frac{1}{2}} U(y)\,dy = 0. \quad (2.7)$$

The solution of this system is easily obtained, being

$$U(y) = \frac{1}{2}\beta Ra\left(\frac{y}{4} - \frac{y^3}{3}\right)$$

$$T(y) = \frac{1}{24}\beta^2 Ra\left(\frac{9y}{80} - \frac{y^3}{2} + \frac{y^5}{5}\right) - y. \quad (2.8)$$

For sufficiently large values of Ra (or β), the solution (2.8) may become unstable, and this possibility is investigated in the next sections.

It is worth pointing out that this type of flow is solely caused by a horizontal density gradient, and exists even when no vertical temperature difference is present ($\Delta T = 0$). This is easily seen by writing (2.8) in dimensional form, setting $\Delta T = 0$.

3. STABILITY ANALYSIS

Following the usual approach of linear stability theory, infinitesimal perturbations (denoted by carets) are introduced into the governing equations. Setting

$$u = U(y) + \hat{u}(x, y, z. t), \; v = \hat{v}(x, y, z, t),$$

$$w = \hat{w}(x, y, z, t) \quad (3.1)$$

$$\theta = T(y) - \beta x + \hat{\theta}(x, y, z, t),$$

$$p = P(x, y) + \hat{p}(x, y, z, t)$$

and neglecting the non-linear terms, we obtain:

$$Pr^{-1}\left(\frac{\partial u}{\partial t} + U(y)\frac{\partial u}{\partial x} + v\,DU(y)\right) = -\frac{\partial p}{\partial x} + \nabla^2 u$$

$$Pr^{-1}\left(\frac{\partial v}{\partial t} + U(y)\frac{\partial v}{\partial x}\right) = -\frac{\partial p}{\partial y} + \nabla^2 v + Ra\theta \quad (3.2)$$

$$Pr^{-1}\left(\frac{\partial w}{\partial t} + U(y)\frac{\partial w}{\partial x}\right) = -\frac{\partial p}{\partial z} + \nabla^2 w$$

$$\frac{\partial\theta}{\partial t} + U(y)\frac{\partial\theta}{\partial x} - \beta u + v\,\mathrm{D}T(y) = \nabla^2\theta \quad (3.3)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \quad (3.4)$$

where the carets have been dropped.

We assume solutions of the form

$$\begin{Bmatrix} u \\ v \\ w \\ \theta \end{Bmatrix} = \begin{Bmatrix} u(y) \\ v(y) \\ w(y) \\ \theta(y) \end{Bmatrix} \exp\left(i(kx + mz - \sigma t)\right) \quad (3.5)$$

for the system (3.2)–(3.4), where the real part is considered to have physical significance. The wave numbers, k and m, are real and the growth rate $\sigma(\equiv\sigma^r + i\sigma^i)$ is complex. Eliminating the pressure from (3.2), and utilizing (3.5), we finally obtain

$$\{Pr(\mathrm{D}^2 - \alpha^2) - ik\beta Ra\overline{U} + i\sigma\}(\mathrm{D}^2 - \alpha^2) \times v + ik\beta Ra\mathrm{D}^2\overline{U}v - \alpha^2 PrRa\theta = 0 \quad (3.6)$$

$$\{\mathrm{D}^2 - \alpha^2 - ik\beta Ra\overline{U} + i\sigma\}\,\theta + \beta u + v - \beta^2 Rav\mathrm{D}\overline{\Theta} = 0 \quad (3.7)$$

$$\{Pr(\mathrm{D}^2 - \alpha^2) - ik\beta Ra\overline{U} + i\sigma\} \times (-\alpha^2 u + ik\mathrm{D}v) + m^2\beta Ra\mathrm{D}\overline{U}v = 0 \quad (3.8)$$

subject to the boundary conditions

$$v = \mathrm{D}^2 v = \mathrm{D}u = \theta = 0 \qquad \text{for} \qquad y = \pm\tfrac{1}{2} \quad (3.9)$$

(for a more detailed derivation of these conditions, see [13]). Here α is the overall wave number defined by $\alpha^2 \equiv k^2 + m^2$. Furthermore, we have introduced

$$\overline{U} \equiv \frac{U(y)}{\beta Ra} = \frac{1}{2}\left(\frac{y}{4} - \frac{y^3}{3}\right)$$

$$\overline{\Theta} \equiv \frac{T(y) + y}{\beta^2 Ra} = \frac{1}{24}\left(\frac{9}{80}y - \frac{y^3}{2} + \frac{y^5}{5}\right). \quad (3.10)$$

Equation (3.6) is the Orr–Somerfeld equation which is coupled with the energy equation (3.7). Equation (3.8) is the vertical component of the vorticity equation combined with the equation of continuity. Except for the last equation, this set has much in common with the one treated by Gill and Davey [14].

4. METHOD OF SOLUTION

To simplify the problem, we shall assume that β is small. Then the equations (3.6)–3.8) may be solved by a perturbation technique using β as a small perturbation quantity. This is a method similar to that introduced in [10]. The solutions are expanded into the form

$$\begin{aligned}(u,v,\theta,k,m,\sigma,Ra) = {} & (u_0,v_0,\theta_0,k_0,m_0,\sigma_0 R_0) \\ & + \beta(u_1,v_1,\theta_1,k_1,m_1,\sigma_1,R_1) \quad (4.1) \\ & + \beta^2(u_2,v_2,\theta_2,k_2,m_2,\sigma_2,R_2) + \\ & \ldots\ldots\end{aligned}$$

where R_i, k_i, m_i, $i = 0, 1, 2, 3\ldots\ldots$ are real quantities and u_i, v_i, θ_i, σ_i generally are complex.

The different orders are obtained by inserting these expressions into (3.6)–(3.8), equating equal powers of β and utilizing the solvability condition. For this procedure to be valid, the Prandtl number must be a zeroth-order quantity.

The zeroth-order system corresponds to thermal convection without shear (the Bénard problem), and the equations are

$$\{Pr(\mathrm{D}^2 - \alpha_0^2) + i\sigma_0\}(\mathrm{D}^2 - \alpha_0^2)v_0 - \alpha_0^2 PrR_0\theta_0 = 0$$

$$\{\mathrm{D}^2 - \alpha_0^2 + i\sigma_0\}\,\theta_0 + v_0 = 0 \quad (4.2)$$

$$-\alpha_0^2 u_0 + ik_0\mathrm{D}v_0 = 0.$$

It is well known [13] that the principle of exchange of stabilities is valid for this system. Hence $\sigma_0^r = 0$. At the neutral state ($\sigma_0^i = 0$), the governing equation may be stated

$$L(v_0) \equiv \{(\mathrm{D}^2 - \alpha_0^2)^3 + \alpha_0^2 R_0\}\,v_0 = 0. \quad (4.3)$$

From (3.9) and (4.2) the boundary conditions may be written.

$$v_0 = \mathrm{D}^2 v_0 = \mathrm{D}^4 v_0 = 0, \qquad y = \pm\tfrac{1}{2} \quad (4.4)$$

The solutions are readily obtained, being

$$v_0 = A\cos\pi y$$

$$u_0 = -i\frac{2k_0}{\pi}A\sin\pi y \qquad (4.5)$$

$$\theta_0 = \frac{2}{3\pi^2}A\cos\pi y$$

and

$$R_0 = \frac{27}{4}\pi^4 \text{ for } \alpha_0^2 = k_0^2 + m_0^2 = \frac{\pi^2}{2}.$$

The amplitude A, which can not be determined from the linear, homogeneous system, may be equated to unity without loss of generality.

Next, for the first-order equations we obtain

$$\begin{aligned}Pr(D^2-\alpha_0^2)^2 v_1 - \alpha_0^2 Pr R_0\theta_1 &= \{2\xi Pr(D^2-\alpha_0^2) \\ &+ ik_0R_0(\overline{U}(D^2-\alpha_0^2) - D^2\overline{U}) \\ &- i\sigma_1(D^2-\alpha_0^2)\}\, v_0 + \{\xi PrR_0 + \alpha_0^2 PrR_1\}\,\theta_0\end{aligned} \qquad (4.6)$$

$$(D^2-\alpha_0^2)\,\theta_1 + v_1 = \{\xi + ik_0R_0\overline{U} - i\sigma_1\} \times \theta_0 - u_0 \qquad (4.7)$$

$$\begin{aligned}Pr(D^2-\alpha_0^2)\{-\alpha_0^2u_1 + ik_0Dv_1\} &= -Pr(D^2-\alpha_0^2)(-\xi u_0 + ik_1Dv_0) \\ &- m_0^2R_0D\overline{U}v_0\end{aligned} \qquad (4.8)$$

where

$$\xi = 2(k_0k_1 + m_0m_1). \qquad (4.9)$$

Eliminating θ_1 from (4.6) by using (4.7), we finally get

$$\begin{aligned}PrL(v_1) = \{&2\xi Pr(D^2-\alpha_0^2)^2 + ik_0R_0(D^2-\alpha_0^2) \\ &\times[\overline{U}(D^2-\alpha_0^2) - D^2\overline{U}] - i\sigma_1(D^2-\alpha_0^2)^2\}v_0 \\ &+ \{\xi PrR_0(D^2-\alpha_0^2) + \alpha_0^2PrR_1(D^2-\alpha_0^2) \\ &+ \alpha_0^2PrR_0\xi + ik_0\alpha_0^2PrR_0^2\overline{U} - i\sigma_1\alpha_0^2PrR_0\} \\ &\times\theta_0 - \alpha_0^2PrR_0u_0\end{aligned} \qquad (4.10)$$

subject to the boundary conditions

$$v_1 = D^2v_1 = D^4v_1 = 0, \qquad y = \pm\tfrac{1}{2}. \qquad (4.11)$$

The operator L has been defined by (4.3).

Under the present conditions, L is easily seen to be self-adjoint. Then a necessary condition for (4.10) to have a solution is that the right hand side be orthogonal to v_0.

Defining the inner-product

$$\langle f, g\rangle \equiv \int_{-\frac{1}{2}}^{+\frac{1}{2}} fg\,dy \qquad (4.12)$$

the condition for solvability may be stated as

$$\langle L(v_1), v_0\rangle = 0. \qquad (4.13)$$

From (4.13) we obtain

$$R_1 = -i\sigma_1\frac{9\pi^2}{2}(1 + Pr^{-1}). \qquad (4.14)$$

Since R_1 must be purely real, this equation gives at the marginal state ($\sigma_1^i = 0$)

$$\sigma_1^r = 0$$

$$R_1 = 0. \qquad (4.15)$$

Thus we have no oscillatory instability to first order. Concerning the Rayleigh number, it is clear from physical considerations that Ra generally can not contain any term involving odd powers of β, since the only effect of changing sign in β, is to reverse the basic velocity, which of course can not alter the stability conditions. Hence

$$R_{2n+1} = 0, \qquad n = 0, 1, 2, \ldots\ldots \qquad (4.16)$$

For the sake of simplicity, we shall represent the basic velocity by a sine-profile, which is indeed a good approximation. This may be seen from Table 1, where the difference between the sine-profile $\overline{U}_1 = \frac{1}{24}\sin\pi y$ and the exact profile $\overline{U}_2 = \frac{1}{2}(y/4 - y^3/3)$ is given in per cent for several values of y.

Table 1. Difference between $\bar{U}_1$ *and* $\bar{U}_2$

y	0	0·1	0·2	0·3	0·4	0·5
Difference in %	0	4·2	3·4	2·1	0·7	0

Throughout the remainder of this analysis, we then take as basic velocity

$$U(y) = \frac{\beta Ra}{24} \sin \pi y . \tag{4.17}$$

The corresponding basic temperature is obtained from (2.5), being

$$T(y) = \frac{\beta^2 Ra}{12\pi^2}(-y + \tfrac{1}{2} \sin \pi y) - y = \Theta(y) - y. \tag{4.18}$$

Introducing $\bar{U} \equiv U/\beta Ra$ and $\bar{\Theta} = \Theta/\beta^2 Ra$ into (4.10) and utilizing $R_1 = \sigma_1 = 0$, we get

$$L(v_1) = ik_0R_0\{\tfrac{3}{64}\pi^4(1 + Pr^{-1}) \sin 2\pi y + \pi \sin \pi y\} \tag{4.19}$$

with boundary conditions $v_1 = D^2v_1 = D^4v_1 = 0$, $y = \pm\frac{1}{2}$.

Setting

$$v_1 = ik_0R_0\tfrac{3}{64}\pi^4(1 + Pr^{-1})\bar{v}_1 + ik_0R_0\pi\tilde{v}_1, \tag{4.20}$$

$\bar{v}_1$ is immediately obtained, while $\tilde{v}_1$ is found most conveniently by Galerkin's method, giving

$$\tilde{v}_1 = \sum_{n=1}^{\infty} A_{2n} \sin 2n\pi y \tag{4.21}$$

where

$$A_{2n} = \frac{64n(-1)^n}{\pi^7(4n^2 - 1)[(8n^2+1)^3 - 27]}.$$

The solution of (4.19) may then be written

$$v_1 = ik_0\{-\frac{3\pi^2}{13.64}(1 + Pr^{-1}) \sin 2\pi y + \pi R_0\tilde{v}_1\}. \tag{4.22}$$

From (4.7) and (4.8) we now obtain the solutions for θ_1 and u_1, namely,

$$\theta_1 = ik_0\left\{-\frac{(27 + Pr^{-1})}{32.39} \sin 2\pi y - \frac{4}{3\pi^3} \sin \pi y + \frac{4}{3\pi^3}\frac{\sinh(\pi y/\sqrt{2})}{\sinh(\pi/2\sqrt{2})} + \frac{2R_0}{\pi}\tilde{\theta}_1\right\} - \frac{4}{9\pi^4}\xi \cos \pi y \tag{4.23}$$

where

$$\tilde{\theta}_1 = \sum_{n=1}^{\infty} \frac{A_{2n}}{8n^2+1} \sin 2n\pi y$$

and

$$u_1 = \pi\left\{\frac{3}{13.16}(1 + Pr^{-1})k_0^2 - \frac{Pr^{-1}}{16}m_0^2\right\} \times \cos 2\pi y - \frac{2k_0^2}{\pi}R_0 D\tilde{v}_1 + \frac{4i}{\pi^3} \times \left\{k_0\xi - \frac{\pi^2}{2}k_1\right\} \sin \pi y - \frac{9\pi Pr^{-1}}{16}m_0^2. \tag{4.24}$$

Here $\tilde{v}_1$ is defined by (4.21).

To determine R_2 and σ_2, we must apply the solvability condition to the second-order equations. These are given by

$$\begin{aligned} Pr\{(D^2 - \alpha_0^2)^2 v_2 - \alpha_0^2 R_0 \theta_2\} &= \{2Pr\xi(D^2 - \alpha_0^2) \\ &+ ik_0R_0[\bar{U}(D^2 - \alpha_0^2) - D^2\bar{U}]\}v_1 + \xi Pr R_0\theta_1 \\ &+ \{2Pr\zeta(D^2 - \alpha_0^2) - Pr\xi^2 + ik_1R_0 \\ &\times [\bar{U}(D^2 - \alpha_0^2) - D^2\bar{U}] - ik_0\xi R_0\bar{U} \\ &- i\sigma_2(D^2 - \alpha_0^2)\}v_0 + \{\zeta Pr R_0 + \alpha_0^2 Pr R_2\}\theta_0 \end{aligned} \tag{4.25}$$

and

$$\begin{aligned} (D^2 - \alpha_0^2)\theta_2 + v_2 &= (\xi + ik_0R_0\bar{U})\theta_1 - u_1 \\ &+ (ik_1R_0\bar{U} + \zeta - i\sigma_2)\theta_0 + R_0 D\bar{\Theta}v_0, \end{aligned} \tag{4.26}$$

where

$$\zeta = k_1^2 + 2k_0k_2 + m_1^2 + 2m_0m_2 . \tag{4.27}$$

The equation for u_2 is not relevant for the present purpose, and is thus not stated.

We now multiply (4.25) and (4.26) by (D^2 −

α_0^2) and $\alpha_0^2 \, PrR_0$, respectively, summing and take the innerproduct with v_0. The solvability condition for this system is then expressed by

$$\langle L(v_2), v_0 \rangle = 0\,. \tag{4.28}$$

After some algebra we obtain from (4.28)

$$R_2 = \frac{3}{16}\pi^2 R_0 + \frac{R_0}{4.13}\left[8 - 8.13\,\Sigma_1 + \frac{81\pi^4}{16.32} + \left(\frac{4}{3} + \frac{3\pi^4}{64}\right)Pr^{-1} + \frac{21\pi^4}{16.32}Pr^{-2}\right]k_0^2 + \frac{7}{3}R_0 Pr^{-1} m_0^2 + 36(k_0 k_1 + m_0 m_1)^2 - i\sigma_2 \frac{9\pi^2}{2}\,(1 + Pr^{-1}), \tag{4.29}$$

where

$$\Sigma_1 = \sum_{n=1}^{\infty} \frac{2\,.\,27\,.\,64n^2}{\pi^4(4n^2-1)^2[(8n^2+1)^3 - 27]}. \tag{4.30}$$

This series converges very rapidly, giving approximately

$$\Sigma_1 = 5 \cdot 63\,.\,10^{-3}. \tag{4.31}$$

At the neutral state, σ_2 must be real. Since R_2 is real, the real and imaginary parts of equation (4.29) reduce, respectively, to

$$R_2 = \frac{3}{16}\pi^2 R_0 + \frac{1}{4.13}R_0\left[8 - 8.13\Sigma_1 + \frac{81\pi^4}{16.32} + \left(\frac{4}{3} + \frac{3}{64}\pi^4\right)Pr^{-1} + \frac{21\pi^4}{16.32}Pr^{-2}\right]k_0^2 + \frac{7}{3}R_0 Pr^{-1} m_0^2 + 36(k_0 k_1 + m_0 m_1)^2 \tag{4.32}$$

and

$$\sigma_2^r \frac{9\pi^2}{2}(1 + Pr^{-1}) = 0. \tag{4.33}$$

The last relation implies that $\sigma_2^r = 0$ at the marginal state, and hence we have no oscillatory instability to second order.

From (4.32) we immediately conclude that R_2 is positive and greater than zero for all kinds of disturbances, which means that the onset of convection in the present problem will occur for a Rayleigh number larger than the critical value corresponding to the Bénard problem. This is to be expected since the basic flow convects warmer fluid in the upper part of the layer and colder in the lower part, thus opposing the destabilizing effect of the temperature difference between the lower and upper plane.

The preferred mode of disturbance will make R_2 a minimum. If we introduce $h = m_0/k_0$ and utilize $k_0^2 + m_0^2 = \pi^2/2$, R_2 may be written

$$R_2(h) = 3/16\pi^2 R_0 + \{A(Pr^{-1}, Pr^{-2}) + B(Pr^{-1}) \times h^2 + 36(k_1 + m_1 h)^2\}\frac{\pi^2}{2(1 + h^2)} \tag{4.34}$$

where the expressions for A and B easily follow from (4.32).

It is then seen that R_2 has an absolute minimum either for $h = 0$, $k_1 = 0$ (transverse rolls) or $h = \infty$, $m_1 = 0$ (longitudinal rolls) depending on the values of A and B, i.e. the Prandtl number.

In Fig. 2, R_2 is displayed for the two kinds of

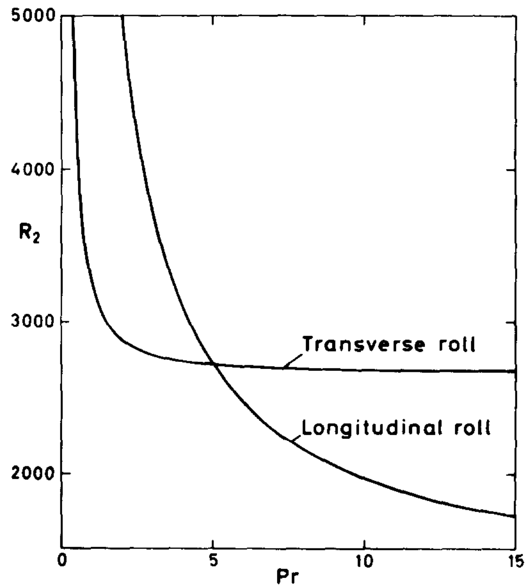

FIG. 2. $R_2 = (Ra - R_0)/\beta^2$ vs Pr for transverse and longitudinal rolls.

rolls, and we observe that for $Pr < 5{\cdot}1$ transverse rolls are preferred, while we get longitudinal rolls when $Pr > 5{\cdot}1$.

It should be noted that formally (4.32) also gives one more point of intersection between the transverse and longitudinal roll curves, namely for $Pr = 0{\cdot}03$. Whether this reflects any real change of mode, however, is doubtful, since our perturbation method is not supposed to be valid at such a small Prandtl number. In fact, we suggest that the tendency to select transverse rolls will be strengthened at small Pr by the increased importance of shear on the mechanism of instability.

When Pr increases towards infinity, the critical Rayleigh number assumes the asymptotic value

$$Ra = R_0(1 + \frac{3}{16}\pi^2\beta^2) + 0(\beta^4). \tag{4.35}$$

5. EXCHANGE OF ENERGY BETWEEN THE MEAN FLOW AND THE PERTURBATION

In this section we shall be concerned with the perturbation energy. Taking the real parts of the component equations in (3.2), multiplying by the real parts of u, v and w, respectively, averaging over a wavelength in the x- and z-direction, adding, and integrating from $y = -\frac{1}{2}$ to $y = +\frac{1}{2}$, using the boundary conditions, we finally obtain the familiar equation for the kinetic energy of the perturbation

$$\frac{1}{2}\frac{\partial}{\partial t}\langle\overline{u^2} + \overline{v^2} + \overline{w^2}\rangle = -\langle \mathrm{D}U(y)\overline{uv}\rangle + PrRa\langle\overline{v\theta}\rangle - Pr\langle(\overline{\nabla u})^2 + (\overline{\nabla v})^2 + (\overline{\nabla w})^2\rangle \tag{5.1}$$

where the bars and the brackets denote mean and vertical integrations, respectively.

Here the term $-\langle \mathrm{D}U\,\overline{uv}\rangle$ represents the conversion of kinetic energy between the perturbation and the mean flow through vertical transfer of horizontal momentum, while the second and third term on the right represent conversion of potential energy to kinetic energy and viscous dissipation, respectively.

We define

$$K \equiv -\langle \mathrm{D}U\,\overline{uv}\rangle. \tag{5.2}$$

Here $\overline{uv} = \frac{1}{2}[u^r(y)v^r(y) + u^i(y)v^i(y)]$ where the superscripts r and i denote real and imaginary parts of the velocities defined by (3.5).

We will consider the marginal stable solutions. Since the solution denoted by subscript zero corresponds to pure convection, it is obvious that $\overline{u_0v_0} = 0$. The lowest order contribution to the Reynolds stresses is then given by

$$\overline{uv} = \beta(\overline{u_1v_0} + \overline{u_0v_1}) + 0(\beta^2). \tag{5.3}$$

For the expression (5.2) we then obtain to second order

$$K = -3{\cdot}6(1{\cdot}6 + Pr^{-1})\beta^2 \tag{5.4}$$

and

$$K = 198{\cdot}1\, Pr^{-1}\beta^2 \tag{5.5}$$

for transverse and longitudinal rolls, respectively. Accordingly, transverse rolls always lose kinetic energy to the mean flow, while longitudinal rolls always gain energy. Similar results were obtained by Asai [15] for convection in Couette flow, from which it was concluded that longitudinal rolls were preferred.

In the present problem a similar conclusion is obviously incorrect. The fastest growing mode will depend on the conversion of potential energy as well as viscous dissipation. This dependence will not fully be explored in this paper. At large Prandtl numbers, however, it is immediately clear from the equation for the kinetic energy (5.1) that the processes mentioned above will dominate. For Pr of about unity, we shall consider one important second order term of the released potential energy.

We define

$$P \equiv PrR_2\,\tfrac{1}{2}\langle v_0\theta_0\rangle\beta^2 = \frac{1}{6\pi^2}PrR_2\beta^2 \tag{5.6}$$

where R_2 is given by (4.32).

From the graph of R_2 in Fig. 2 it follows that the value of P for transverse rolls will be less than its value for longitudinal rolls when Pr is less than five, while for Pr greater than five the opposite is true. Further it can be shown that for $Pr = 5$ the release of potential energy (P) for a longitudinal roll is about six times larger than the energy converted from the mean flow through vertical momentum transfer. This indicates that the process of conversion of potential energy will dominate for Pr about unity, and may account for the change of mode at $Pr = 5{\cdot}1$. Since we consider marginally stable solutions, the left hand side of (5.1) is zero. To satisfy this condition, the viscous dissipation must also be important for Pr of about unity.

6. SUMMARY AND DISCUSSION

When the Prandtl number is less than 5·1, we find that the Rayleigh number at the neutral state has a minimum for steady, transverse rolls, i.e. rolls with axes normal to the mean flow. For Prandtl numbers greater than 5·1, the Rayleigh number is smallest for steady, longitudinal rolls having axes aligned in the direction of the mean flow.

Our conclusions are, in some respects, similar to those reached by Liang and Acrivos [10] for convection in a tilted slot. As in the present case, the neutral state remains stationary for all disturbance wave numbers, i.e. the principle of exchange of stabilities applies, and the critical Rayleigh number decreases with increasing Pr to an asymptotic value independent of Pr. In the present problem this limit is given by (4.35).

Two important differences may be noted, however. In [10] the most unstable mode was found to be a longitudinal roll, and the critical Rayleigh number the same as for pure convection without shear. In our case, the most unstable mode may be either transverse or longitudinal depending on whether Pr is smaller than 5·1 or not. The critical Rayleigh number will always be larger than that corresponding to convection without horizontal density gradients. Physically this is due to the upward convection of warm fluid and downward convection of cold fluid in the basic flow.

The last section has been devoted to energy considerations. We have shown that, analogous to [15], a longitudinal roll always gains kinetic energy from the mean flow through vertical transfer of horizontal momentum, while a transverse roll always loses energy by this process. This does not explain the change of mode at $Pr = 5{\cdot}1$ in the present problem. By computing one particular term in the released potential energy, it is indicated that the mechanism of instability is primarily of convective origin.

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CONVECTION THFRMIQUE ENTRE DES PLANS CHAUFFES NON UNIFORMEMENT

Résumé—On a étudié sur la base d'une théorie linéaire la stabilité de la convection naturelle dans une mince couche horizontale soumise à des gradients de température aussi bien horizontaux que verticaux. Les limites sont sans contrainte et parfaitement conductrices et le gradient de température horizontal est supposé petit. L'analyse montre que le nombre critique de Rayleigh est toujours plus grand que celui du problème classique de Bénard. Le mode préféré de perturbation est stationnaire et sera un rouleau transversal (avec axe normal à l'écoulement fondamental) ou un rouleau longitudinal (avec un axe dans la direction de l'écoulement fondamental) selon que le nombre de Prandtl est inférieur ou supérieur à 5,1. Finalement, on a fait quelques calculs sur l'énergie convertie associée aux perturbations instables et qui indiquent que le mécanisme d'instabilité est d'origine thermique (par convection).

ÜBER THERMISCHE KONVEKTION ZWISCHEN UNGLEICHMÄSSIG BEHEIZTEN PLATTEN

Zusammenfassung—Die Stabilität der natürlichen Konvektion in einer dünnen, horizontalen Schicht, die sowohl vom horizontalen als auch vom vertikalen Temperaturgradienten abhängig ist, wird auf der Basis der linearen Theorie untersucht.

Die Grenzen werden so gewählt, dass sie spannungsfrei und vollkommen leitend sind; vom horizontalen Temperaturgradienten wird vorausgesetzt, dass er klein ist. Die Analyse zeigt, dass die kritische Rayleigh-Zahl immer grösser ist als die für das gewöhnliche Benard-Problem. Die bevorzugte Störungsart ist stationär; sie wird ein Querwirbel (Achsen senkrecht zur Grundströmung) oder ein Längswirbel (Achsen in Richtung der Grundströmung) sein, je nach dem die Prandt-Zahl kleiner oder grösser 5,1 ist. Schliesslich werden einige Berechnungen für die übertragene Energie angestellt, die mit den instabilen Störungen verknüpft ist. Das gibt den Hinweis, dass der Mechanismus der Instabilität von thermischem (konvektivem) Ursprung ist.

О КОНВЕКЦИИ ТЕПЛА МЕЖДУ НЕРАВНОМЕРНО НАГРЕВАЕМЫМИ ПЛОСКОСТЯМИ

Аннотация—На основе линейной теории исследуется устойчивость естественной конвекции в тонком горизонтальном слое, подвергаемом воздействию горизонтальных, а также вертикальных градиентов температуры. Принято, что напряжение на границах отсутствует, границы обладают совершенной теплопроводностью и горизонтальный температурный градиент мал. Путём анализа показано, что критическое значение числа Релея всегда больше значения для обычной задачи Бенарда. Из способов возмущения выбран стационарный, и в зависимости от того, больше или меньше 5,1 значение числа Прандтля, источником возмущения служат поперечные (с осями, расположенными перпендикулярно к основному потоку) или продольные вихри (с осями, направленными вдоль основного потока). Наконец, выполнены некоторые расчёты обращенной энергии, связанной с неустойчивыми возмущениями, которые показывают, что механизм неустойчивости имеет термическое (конвективное) происхождение.